Four equivalent properties of integrable billiards

Hamiltonian Systems Seminar, November 2020

A. Glutsyuk, I. Izmestiev, S. T., arXiv:1909.09028, to appear in Israel J. Math.

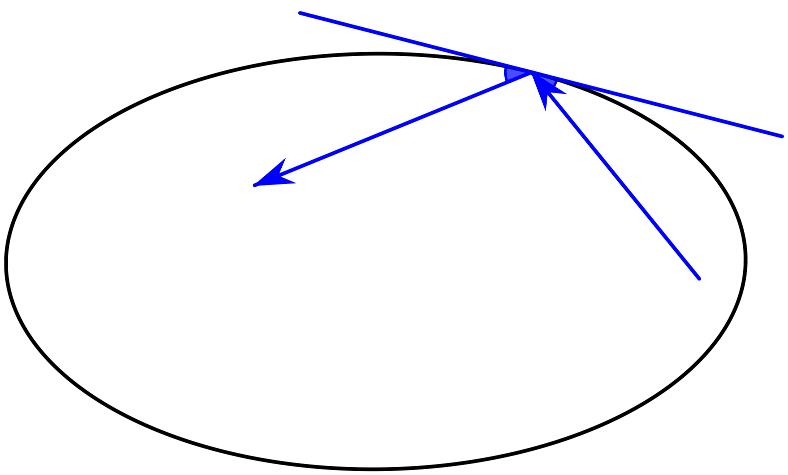
A. Glutsyuk. On curves with Poritsky property. arXiv:1901.01881

I. Izmestiev, S. T. Ivory's theorem revisited. J. Integrable Syst. 2 (2017)

M. Arnold, S. T. Remarks on Joachimsthal integral and Poritsky property. arXiv:2009.04988

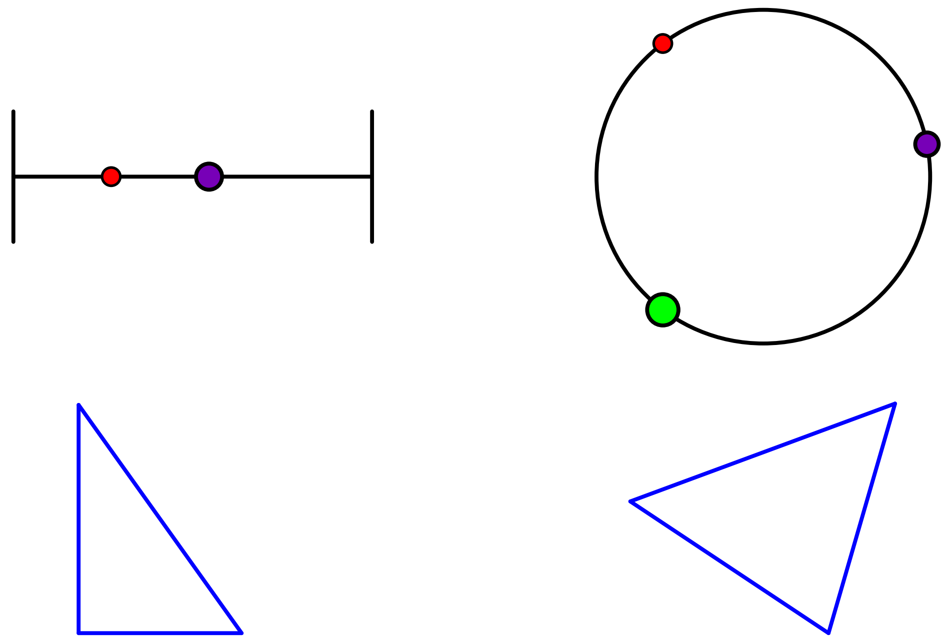
Why billiards?

Motion of a free mass point with elastic reflection off the boundary.



Geometrical (ray) optics: ideal mirror reflection.
Mechanical systems with elastic collision (preserving energy and momentum), including ideal gas models.

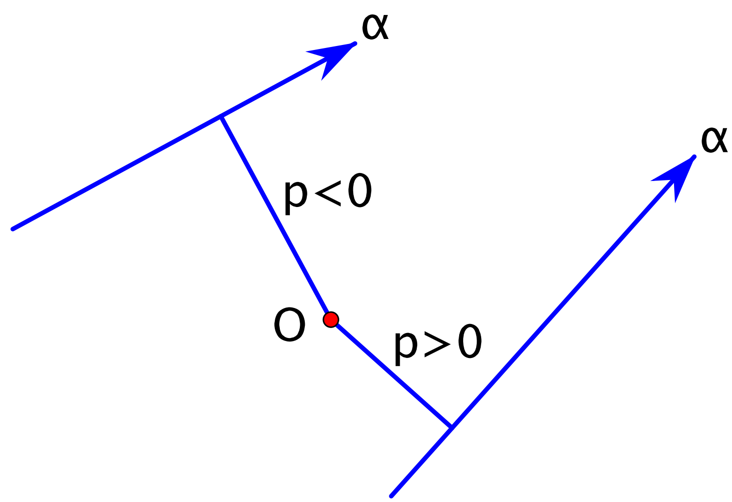
A simple example:



$$\alpha_i = \text{arctan}\left(m_i\sqrt{\frac{m_1+m_2+m_3}{m_1m_2m_3}}\right), \quad i = 1, 2, 3.$$

An obtuse triangle?

Symplectic structure on oriented lines (rays of light)

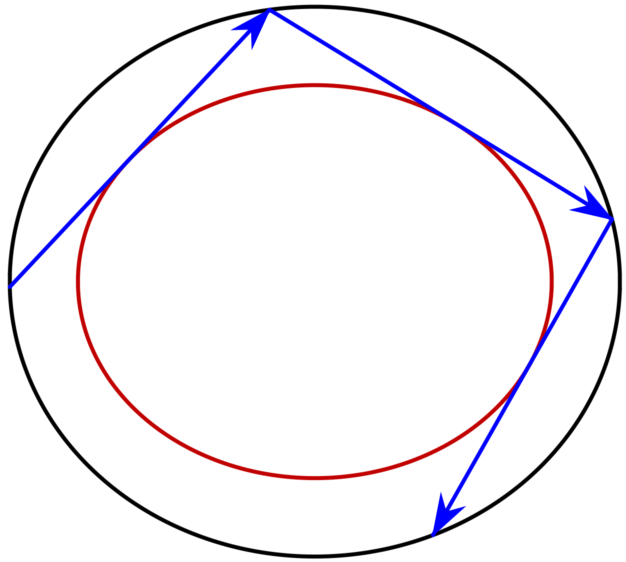


$$\omega = dp \wedge d\alpha.$$

Likewise for oriented geodesics of a Riemannian manifold (via symplectic reduction from the cotangent bundle). The optical (billiard) reflection is a symplectic map.

Which symplectic maps can be realized by optical systems?

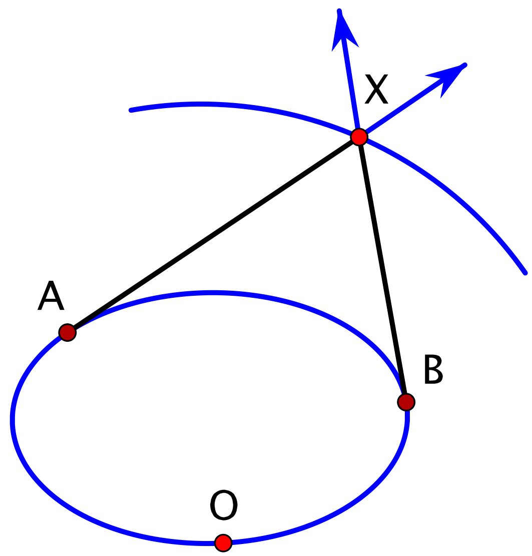
Caustics



Existence: if the billiard is strictly convex and sufficiently smooth (KAM theory, Lazutkin's theorem, 1973).

But they are impossible to contstruct.

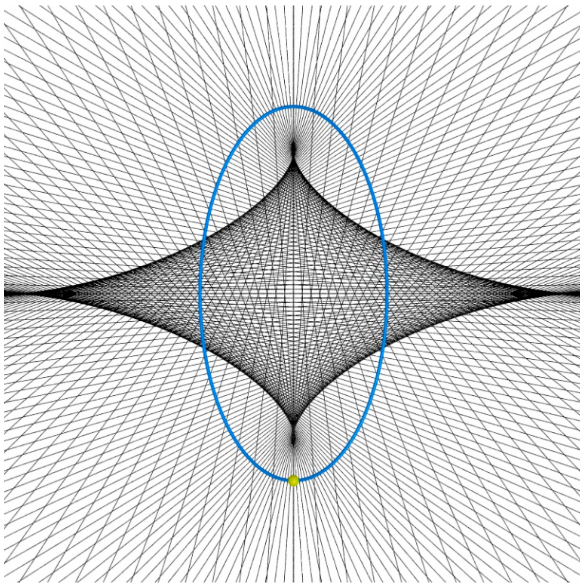
String construction

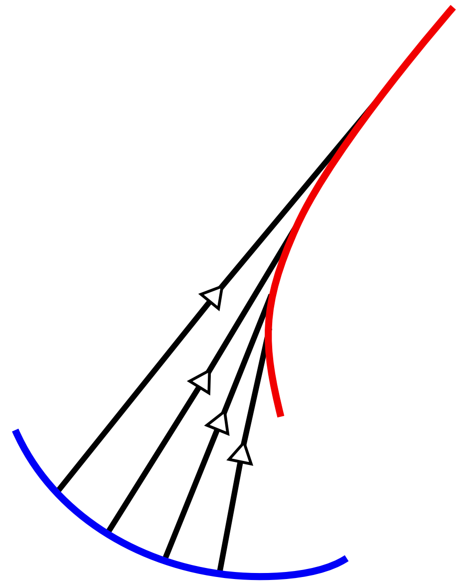


$$\Gamma = \{X : |XA| + |\overset{\smile}{AO}| + |XB| + |\overset{\smile}{BO}| = const\}.$$

Proof: $\nabla(|XA| + |\overset{\smile}{AO}|)$ and $\nabla(|XB| + |\overset{\smile}{BO}|)$ are unit vectors along AX and BX, their sum is orthogonal to Γ.
This yields the *string diffeomorphisms* $A \mapsto B$.

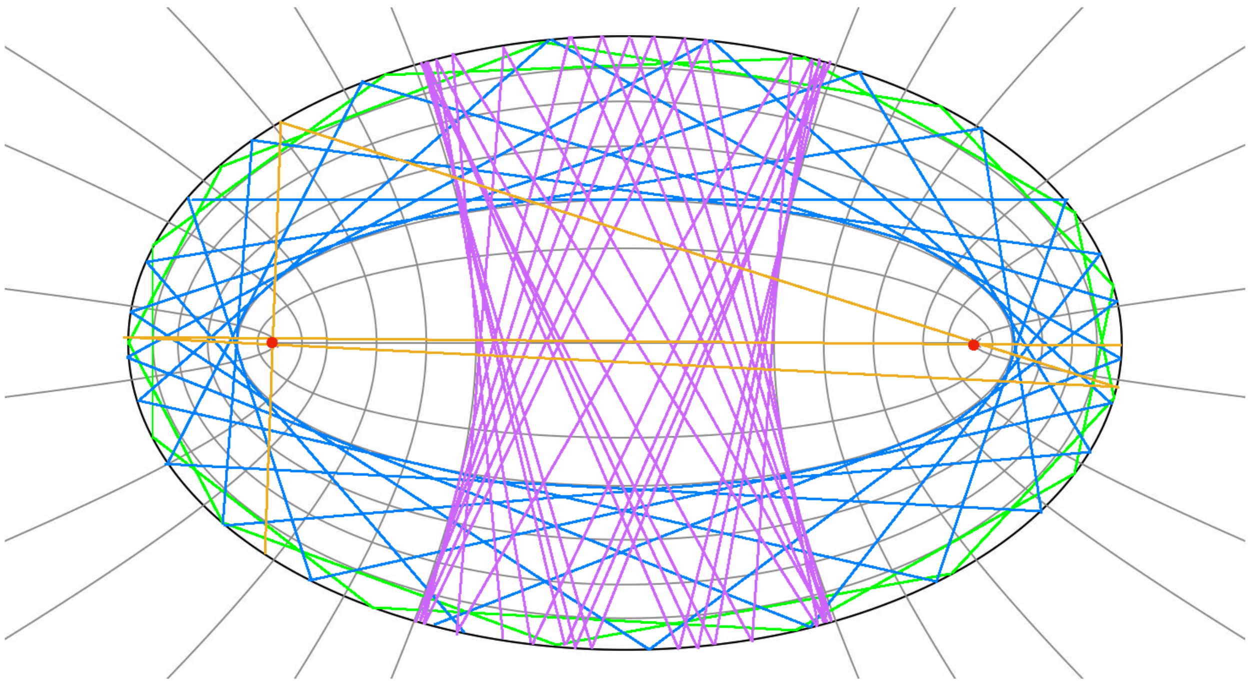
Comparison: evolutes and involutes





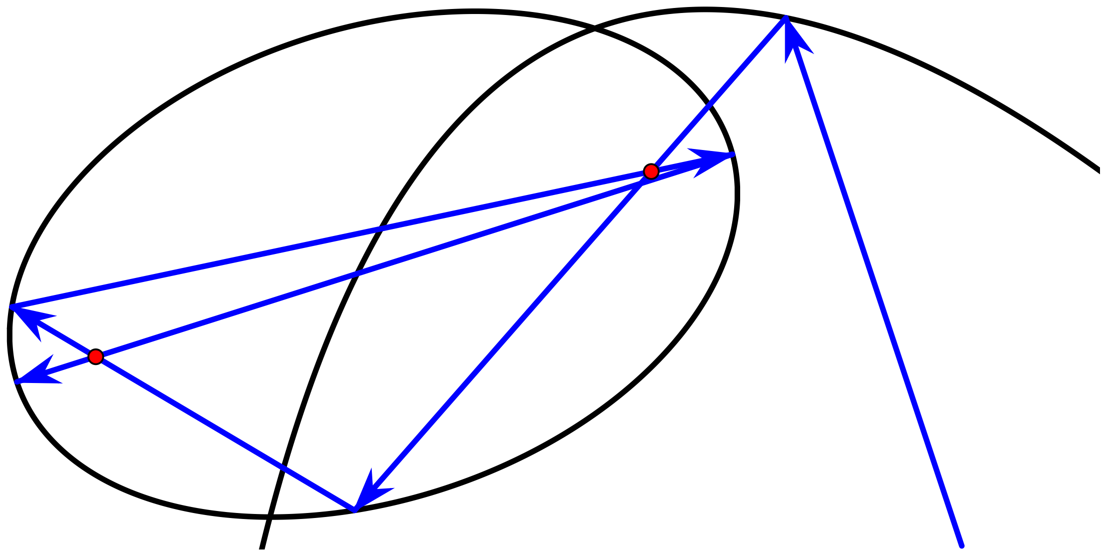
Evolute: the envelope of the normals. *Involute*: given by string construction; come in 1-parameter families.

Billiard in ellipse



Caustics: the confocal conics. The Graves theorem: *the string construction of an ellipse yields a confocal ellipse*.

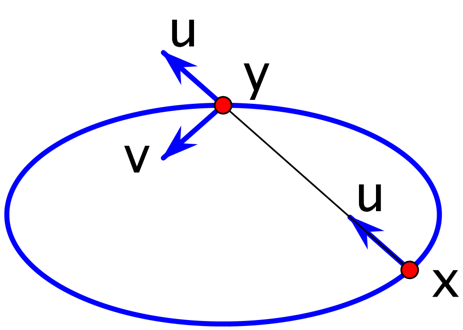
Trap for a parallel beam of light:

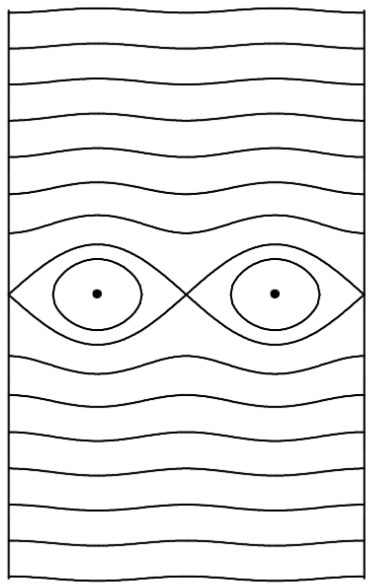


But one cannot trap a 2-parameter family of rays, a consequence of the Poincaré recurrence theorem.

In dimension d, how much light can one trap?

Phase space, phase portrait, and the Joachimsthal integral:





If the ellipse is $(Ax, x) = 1$, then

$$(Ax, u) = -(Ay, u) = (Ay, v).$$

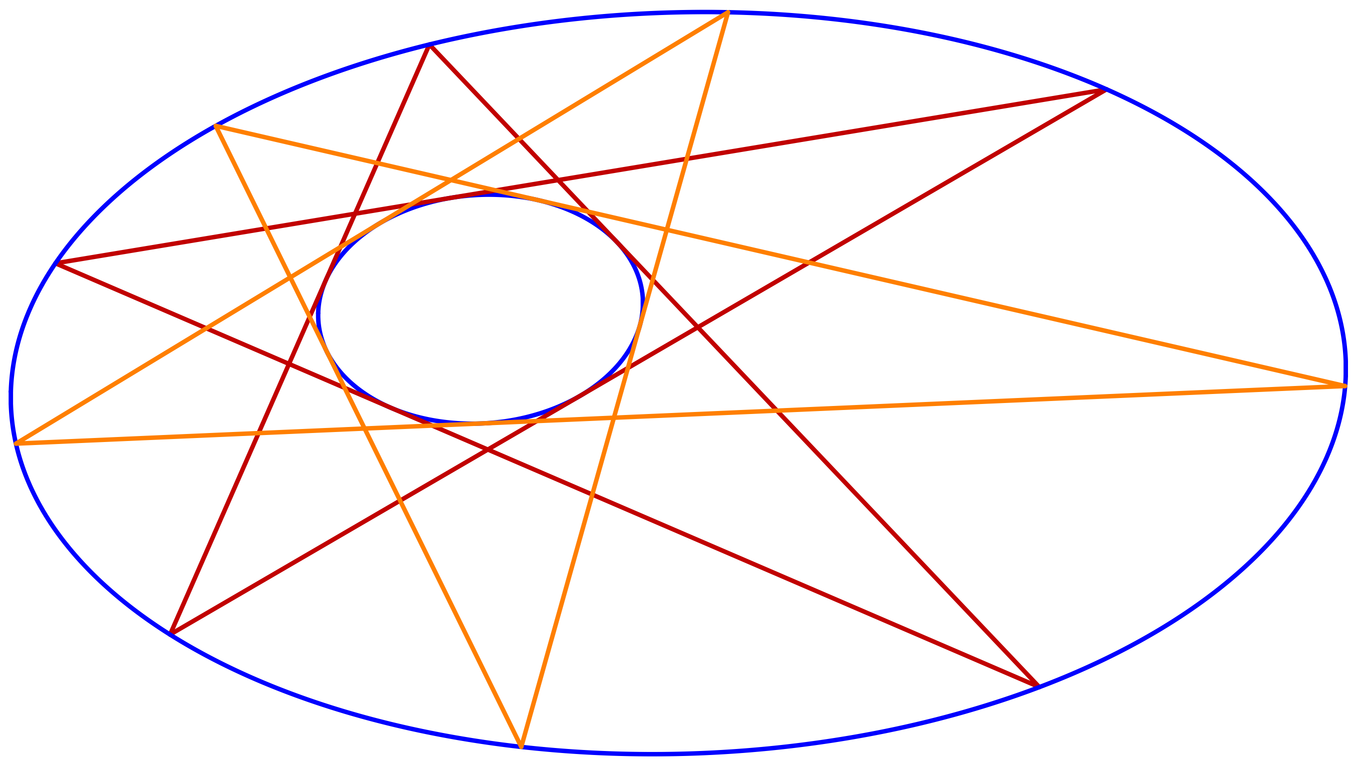
Birkhoff-Poritsky Conjecture: *the only billiards integrable near boundary are the elliptic ones.*

Recent progress: rigidity of circles (Bialy); perturbative versions (Avila, De Simoi, Kaloshin, Sorrentino); algebraic integrability (Bialy, Mironov, Glutsyuk).

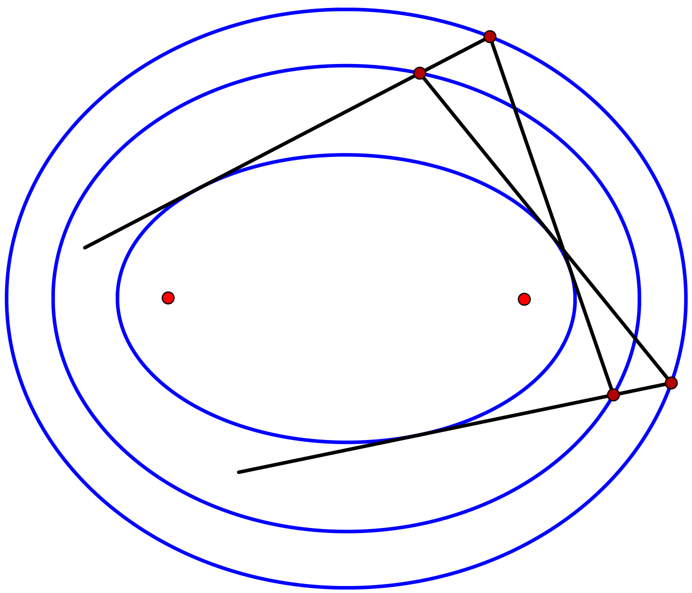
Consequence of integrability (Arnold-Liouville theorem): a special (Poritsky) parameter t on every ellipse.
If F is an integral, then d/dt is the Hamiltonian vector field of F.
The 1-form dt is invariant, and the billiard maps, i.e., the string diffeomorphisms, are shifts: $t \mapsto t + c$.

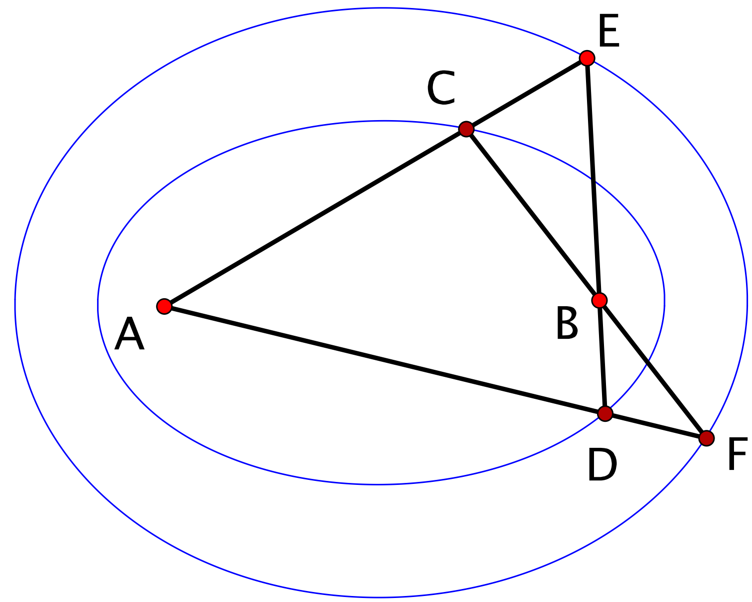
Corollaries

Poncelet porism:



The reflections from confocal ellipses commute:

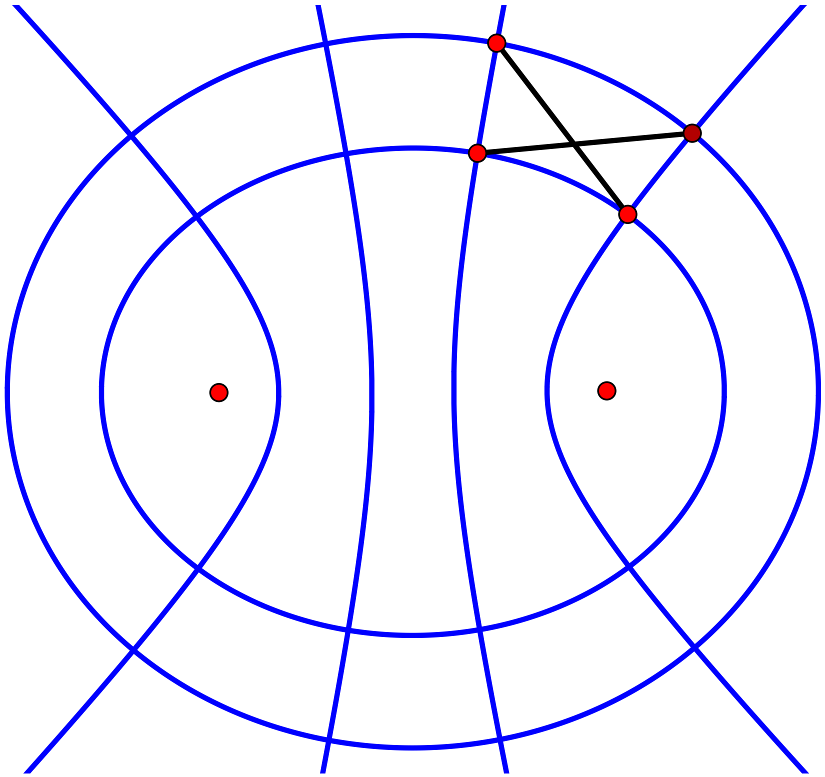




"The most elementary problem of elementary geometry" (Pedoe):

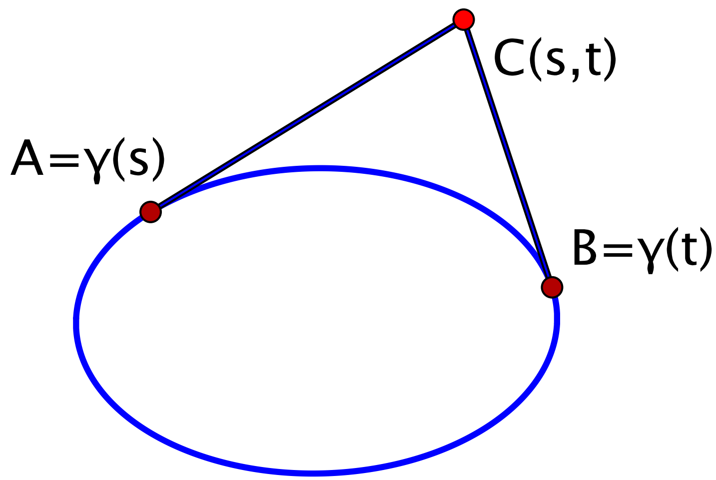
$$AC + CB = AD + DB \iff AE + EB = AF + FB$$

Ivory's lemma



On the attraction of homogeneous ellipsoids. Phil. Trans. Royal Soc. London **99** (1809), 345–372.

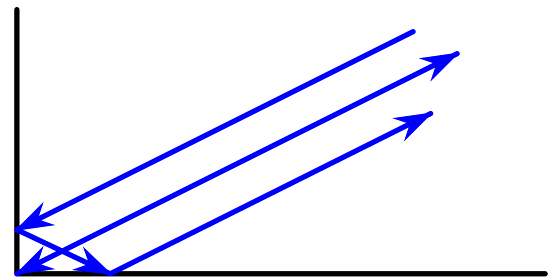
Coordinates outside of an ellipse:



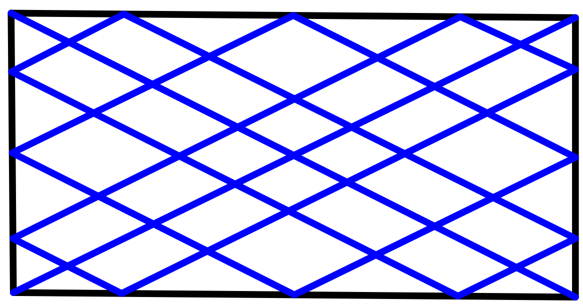
The confocal ellipses are $\{t - s = \text{const}\}$, the confocal hyperbolas are $\{t + s = \text{const}\}$. The billiard reflection in a confocal ellipse is $t \mapsto t + c$, and in a confocal hyperbola $t \mapsto c - t$.

Ivory's lemma by way of billiards

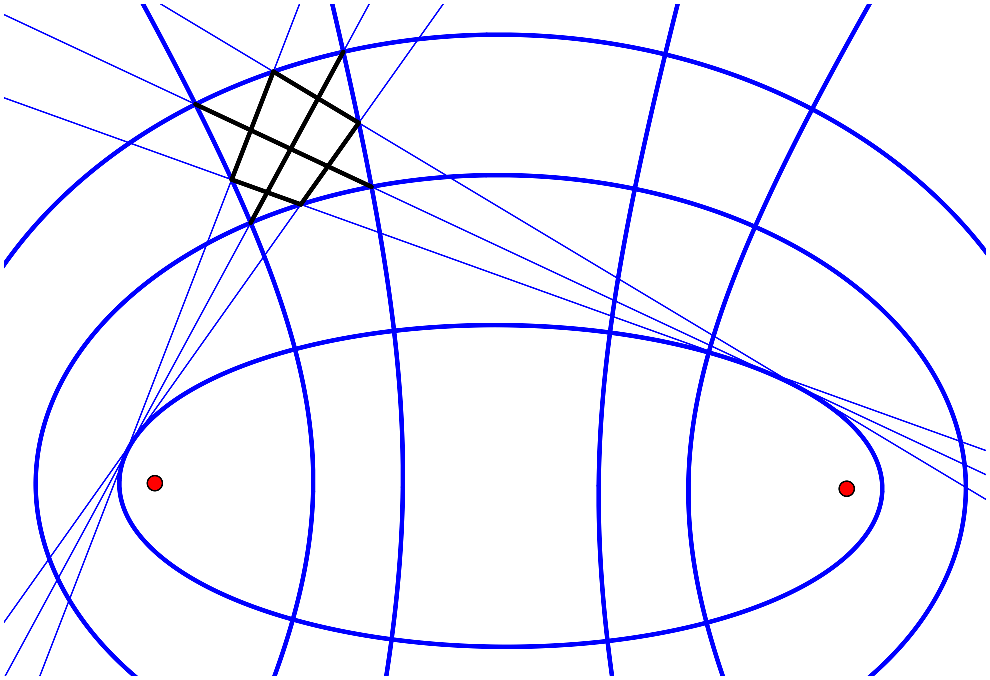
Corner reflector:



Diagonal of a rectangle are equal:

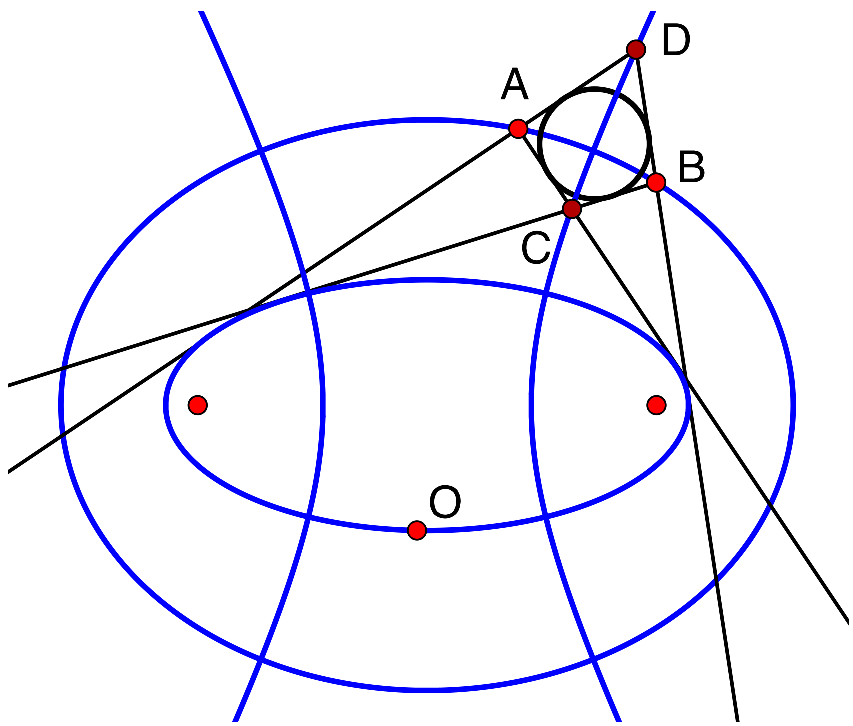


Likewise for confocal conics:



The composition of four reflections is a shift with a fixed point (a diagonal), hence it is the identity.

Chasles-Reye theorem



Coordinates s_1, s_2, t_1, t_2. Then $t_1 - s_1 = t_2 - s_2$, hence $t_1 + s_2 = t_2 + s_1$, and therefore points C and D lie on a confocal hyperbola.

Let f and g be the distances from points to point O going around the ellipse. Then

$$f(A)+g(A)=f(B)+g(B), \quad f(C)-g(C)=f(D)-g(D),$$

hence

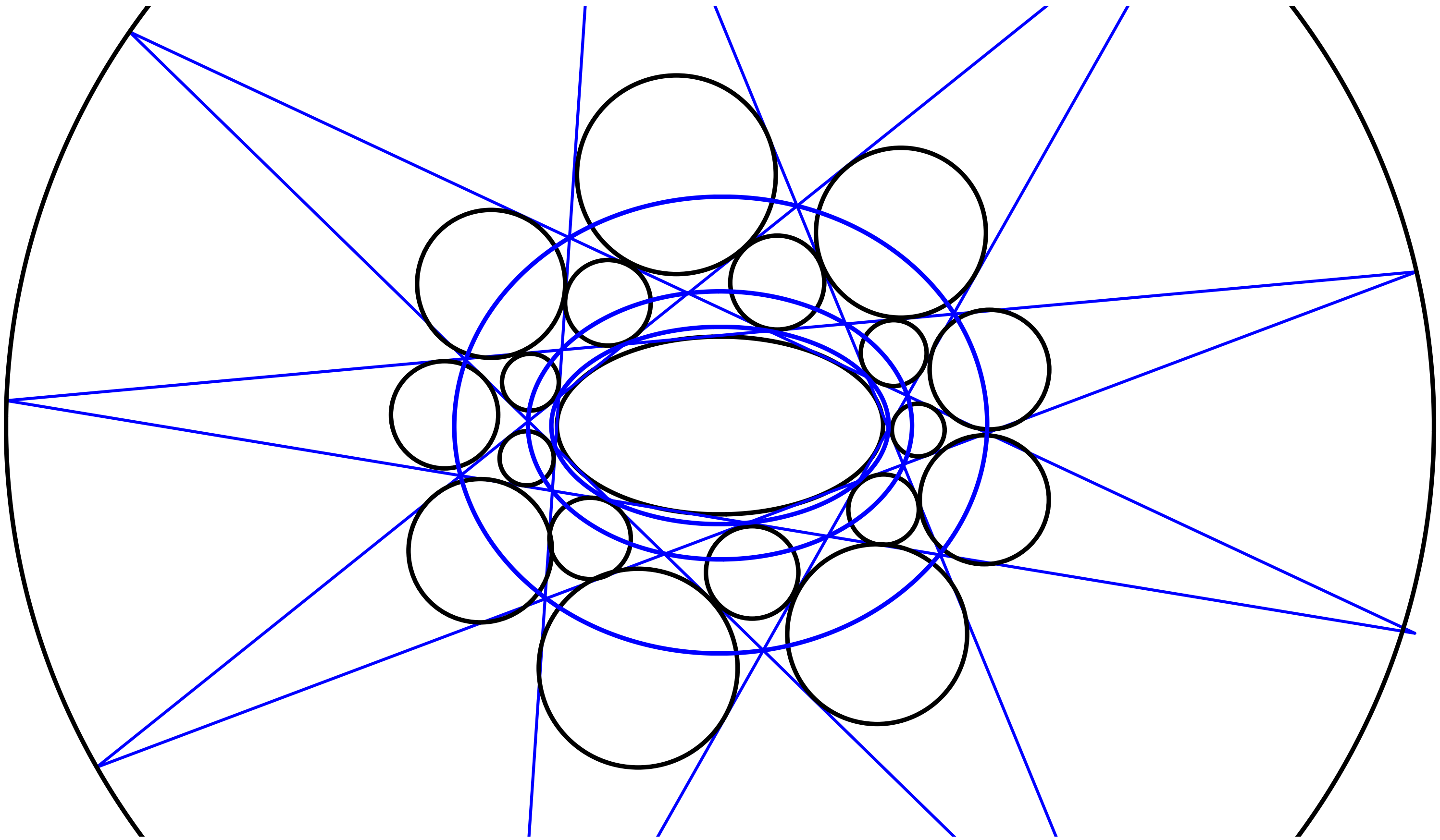
$$f(D)-f(A)-g(A)+g(C)+f(B)-f(C)-g(D)+g(B)=0,$$

or

$$|AD|+|BC|=|AC|+|BD|,$$

and the quadrilateral is circumscribed.

Poncelet grid of circles



Back to Poritsky parameter.

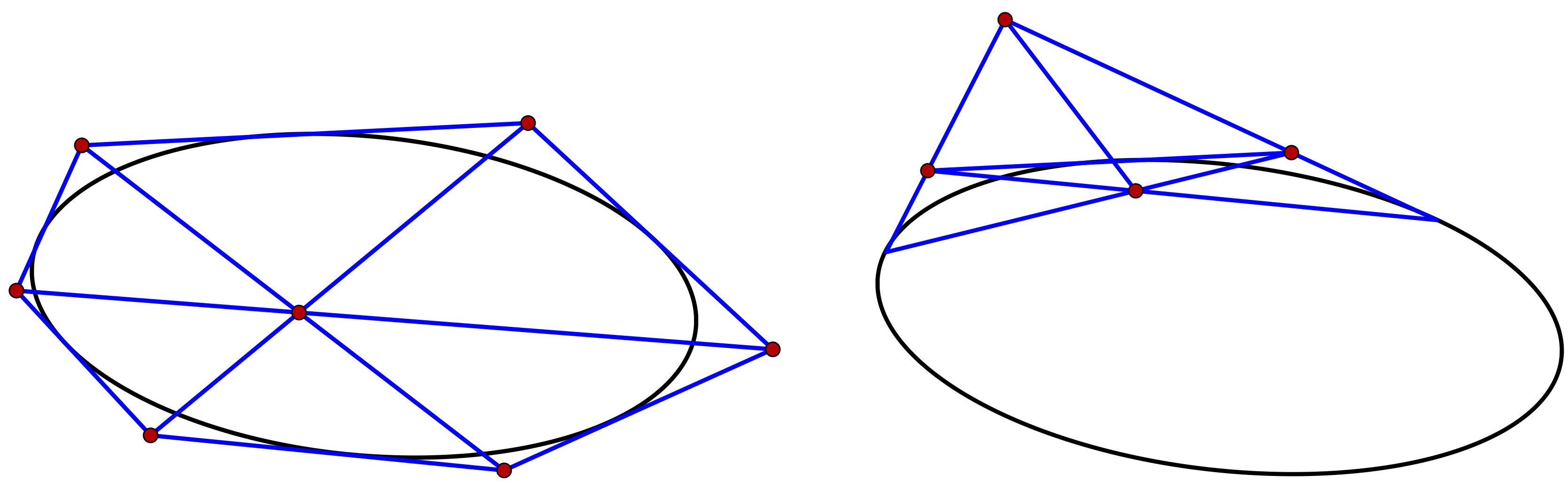
Poritsky theorem [1950]: *A (germ of a) curve in the Euclidean plane that possess a Poritsky parameter is a conic*.

Extended to spherical and hyperbolic geometries, and to outer billiards, by A. Glutsyuk. For outer billiards, an analog of the string construction is the *area construction*.

The relation to the arc length parameter s is $dt = k^{2/3}\ ds$ (for outer billiards, it's $dt = k^{1/3}\ ds$, the affine length element). This is how, in the limit, the impact points of any billiard, not necessarily elliptic, are distributed.

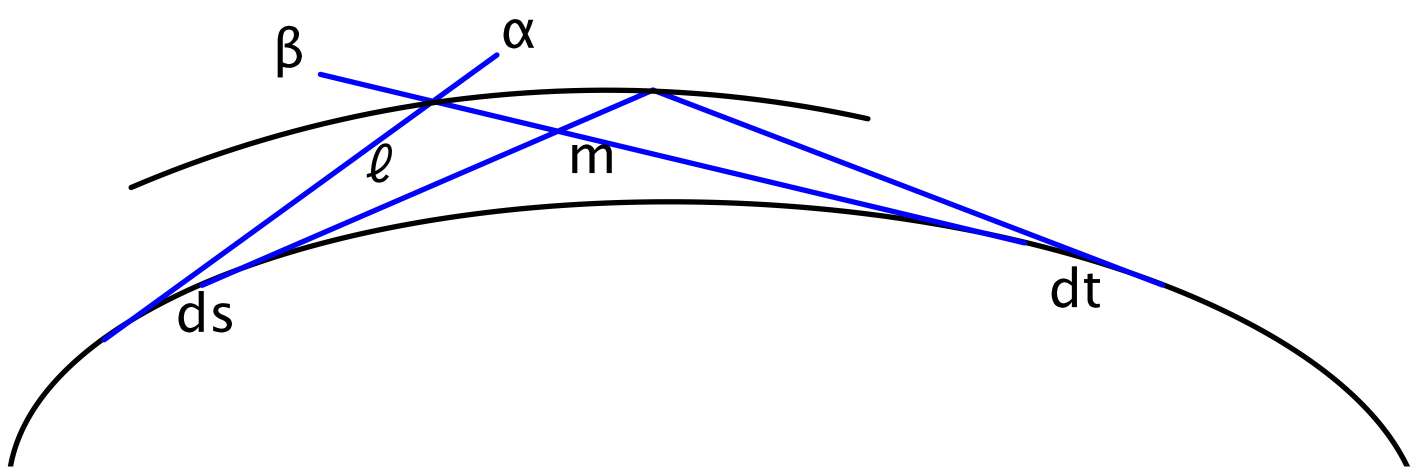
Sketch of proof of Poritsky theorem

Brianchon theorem:



The converse holds as well.

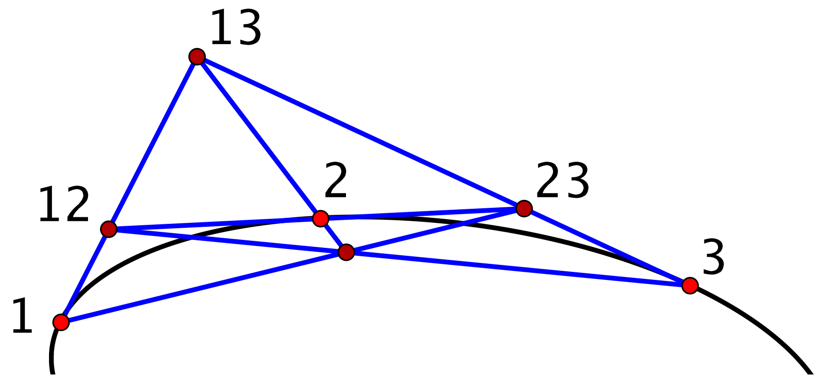
One has $\ell d\alpha = m d\beta$ (always) and $ds = dt$ (Poritsky property).



Also $s(\alpha)$ and $t(\beta)$, hence $s' d\alpha = t' d\beta$, and then

$$\frac{m}{\ell} = \frac{d\alpha}{d\beta} = \frac{t'}{s'}.$$

Therefore



$$\frac{|2,12|}{|1,12|} = \frac{t'_2}{t'_1}, \; \frac{|3,23|}{|2,23|} = \frac{t'_3}{t'_2}, \; \frac{|1,31|}{|3,31|} = \frac{t'_1}{t'_3},$$

hence

$$\frac{|2,12|}{|1,12|}\frac{|3,23|}{|2,23|}\frac{|1,31|}{|3,31|} = 1,$$

and Ceva's theorem implies that the lines are concurrent. *QED*

Three properties

1. Graves property (of an annulus foliated by convex closed curves): a caustic of a caustic is a caustic;

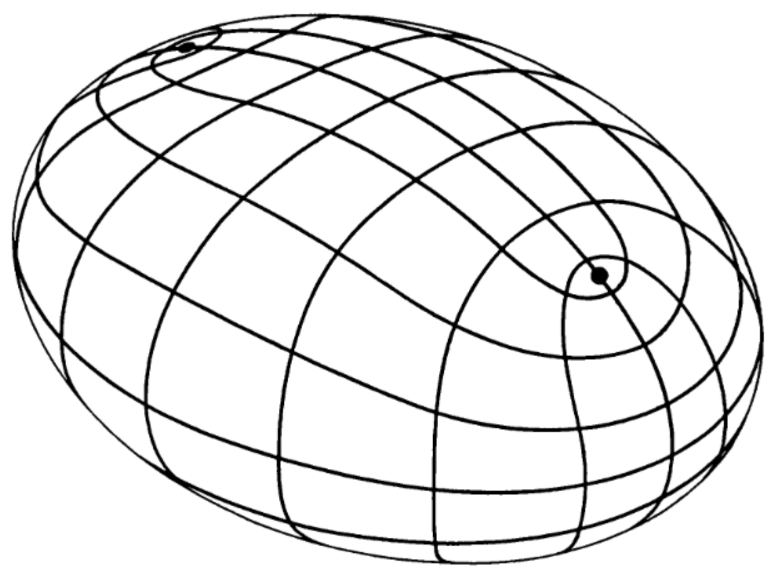
2. Poritsky property (of a strictly convex curve): the string diffeomorphisms are shifts;

3. Ivory property (of a net – or a 2-web – of curves).

And, as we saw, (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Other integrable billiards

Conics in S^2 and H^2, and ellipsoids in $\mathbf{R}^3$:



The three properties hold for the lines of curvature (which are the intersections with confocal quadrics).

Elliptic coordinates and Liouville metrics

For confocal family of conics

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}=1,$$

one has the elliptic coordinates (λ,μ), and

$$dx^2+dy^2=(\lambda-\mu)\left(\frac{d\lambda^2}{4(a+\lambda)(b+\lambda)}-\frac{d\mu^2}{4(a+\mu)(b+\mu)}\right).$$

More general, Liouville metrics:

$$ds^2=(U_1(u)-V_1(v))\ (U_2(u)du^2+V_2(v)dv^2),$$

and Liouville nets of coordinate curves $u=const$ and $v=const$.

Example: lines of curvature on an ellipsoid.

The geodesic flow of a Liouville metric is integrable by separation of variables; it has an integral, quadratic in momentum.

Consider an annulus $\mathcal{A}$ with a Riemannian metric and a foliation $\mathcal{F}_1$ by smooth geodesically convex curves. Let $\mathcal{F}_2$ be the foliation by the orthogonal curves.

Theorem: *The following four properties are equivalent:*
(i) The foliation $\mathcal{F}_1$ has the Graves property;
(ii) The inner boundary curve of $\mathcal{A}$ has the Poritsky property;
(iii) The foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ form a Liouville net;
(iv) The net $(\mathcal{F}_1, \mathcal{F}_2)$ in $\mathcal{A}$ has the Ivory property.

There is a local version of this result as well.

Generalized Birkhoff-Poritsky conjecture: *Given an annulus with a Riemannian metric in which one of the components* Γ *of the boundary is strictly convex, consider the billiard system near this component. If a neighborhood of* Γ *is foliated by caustics, then the metric near* Γ *is Liouville, and* Γ *is a coordinate line.*

This implies Birkhoff's conjecture, due Weihnacht's classification of Liouville nets in $\mathbf{R}^2$ (1924):

Theorem: *Liouville nets are of one of the following types:*
a) Confocal ellipses and hyperbolas;
b) Confocal and coaxial parabolas;
c) Concentric circles and their radial lines;
d) Two families of orthogonal lines.

About proofs of the main theorem

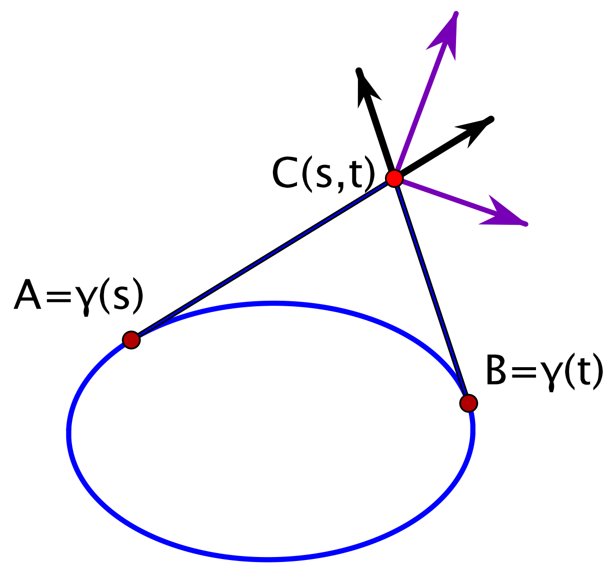
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) were sketched.

That (iii) $\Leftrightarrow$ (iv), i.e., Ivory is equivalent to Liouville, is a theorem of Blaschke and Zwirner (1927-28). See also I. Izmestiev and S. T. *Ivory's Theorem revisited*.

That (iii) $\Leftrightarrow$ (i), i.e., Liouville is equivalent to Graves, is due to Darboux: *Leçons sur la Théorie générale des Surfaces et les Applications géométriques du Calcul infinitésimal*. Troisiéme partie, 1894, item 589, Livre VI, Chap. I.
See also V. Dragović, M. Radnović. *Poncelet porisms and beyond.* Birkhäuser/Springer, 2011.

Finally, (ii) $\Rightarrow$ (iii), i.e., Poritsky implies Liouville.

Recall the coordinates (s,t) near γ on a Riemannian surface:



Lemma: *The coordinates* $x = (s+t)/2, y = (t-s)/2$ *are orthogonal, and the diagonals* $x \pm y = const$ *are geodesics.*

And a general result:

Theorem: *If a Riemannian metric, written in orthogonal coordinates* (x, y), *has the property that the diagonals* $x \pm y = const$ *are geodesics, then this metric is Liouville.*

The proof is computational, and I do not dwell on it.

Joachimsthal integral revisited

Theorem: *Assume that a convex curve γ admits a non-vanishing normal vector field N such that for every points $\gamma(x), \gamma(y)$, one has*

$$N(x) \cdot (\gamma(y) - \gamma(x)) = -N(y) \cdot (\gamma(y) - \gamma(x)).$$

Then γ is a conic.

The first step of the proof is that γ has the outer Poritsky property: the segments $[\gamma(x), \gamma(y)]$ with $y - x = c$ cut off constant areas.

The theorem also holds in the spherical and hyperbolic geometries, and in the higher dimensional case as well.

In the multi-dimensional case, this follows from the result that, essentially, is due to M. Berger:
If every transverse 2-dimensional section of a smooth hypersurface in the Euclidean space is a (part of a) conic, then the hypersurface is a (part of a) quadric.

Thank you!